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An Extension to Mairhuber's Theorem. On Metric Projections and Discontinuity of Multivariate Best Uniform Approximation

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1. INTRODUCTION

Throughout the paper X will denote a compact Hausdorff space and $C(X)$ the Banach space, with the uniform norm, of real continuous functions on X. The distance of a function f from a subset M of $C(X)$ is

$$d(f, M) = \inf\{\|f - p\|: p \in M\},$$

and the set

$$P(f) = \{p \in M: \|f - p\| = d(f, M)\}$$

is the set of best uniform approximations to f from M. The set-valued mapping P is the metric projection of $C(X)$ onto M. A continuous mapping $s: C(X) \to M$ such that $s(f) \in P(f)$ for all $f \in C(X)$ is known as a continuous selection for the metric projection, but for brevity will also be referred to here as simply "a continuous selection for M." Continuous selections for metric projections have been considered by a number of authors (see [1, 3–5, 9, 10] and references given in these papers).

This paper is concerned with linear subspaces M of $C(X)$ with the property that no non-zero function in M has the value zero at all points of some non-empty open subset of X—following [3] and [1] such M will be called Z-subspaces of $C(X)$. If M is a Z-subspace of $C(X)$, of dimension at least two, then X can have no isolated points. The principal result of the paper shows that there can be a continuous selection for a finite-dimensional Z-subspace M only in very limited situations. In order to state the result precisely some further terminology must be introduced.

A finite-dimensional linear subspace M of $C(X)$ is said to be Chebyshev if

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$P(f)$ is a singleton for each $f \in C(X)$, and of Chebyshev rank k (or k-Chebyshev) if $\dim P(f) \leqslant k$ for each $f \in C(X)$.

Certain non-metrisable topological spaces enter into the discussion. Let $I = [0, 1]$ and let A be any subset of I. We will write $I_A = I \times \{0\} \cup A \times \{1\}$. Let $\leqslant$ be the lexicographic order on I_A: $(s, \sigma) \leqslant (t, \tau)$ if and only if $s < t$ or $s = t$ and $\sigma \leqslant \tau$. The ordering is linear and I_A is order complete in the ordering. Let I_A be given the order topology. Then I_A is a separable compact Hausdorff space. It can be thought of as the interval I with each point of the subset A "split" into two points so as to introduce a gap. A space constructed in this way will be called an interval with split points. In the case $A = I$ the space is often known as "the split interval." It will be shown later that I_A is metrisable if and only if A is countable. Note also that $I_\varnothing$ is homeomorphic to I.

The main results can now be stated:

THEOREM. *Suppose that X is a compact Hausdorff space and that there exists a Z-subspace M of $C(X)$, of finite dimension at least two, such that there is a continuous selection for the metric projection onto M.*

If X is metrisable then X is homeomorphic to a subspace of a circle. If X is not metrisable then X is homeomorphic to a subspace of an interval with split points, and M is 1-Chebyshev but not Chebyshev.

Mairhuber's theorem asserts that if there exist Chebyshev subspaces of $C(X)$, of finite dimension not less than two, then X is homeomorphic to a subspace of a circle. If M is finite dimensional and Chebyshev then the metric projection (regarded as a mapping of $C(X)$ onto M) is continuous. Therefore, for Z-subspaces M, the theorem is an extension of Mairhuber's theorem. References to the literature concerning Mairhuber's theorem and also the proof due to Schoenberg and Yang [7] can be found in [8].

Perhaps the most interesting aspect of our theorem is its implication for multivariate approximation. If X is, for example, a square then there can be no continuous selection for any Z-subspace M of finite dimension greater than one. This must imply that any algorithm for the computation of best uniform approximations from M must be in some sense unstable. It might be of interest to know if there is numerical experience which corresponds to this fact.

The proof of the theorem follows from a number of lemmas. The formulation and proof of Lemma 3 originated in a consideration of the arguments used by Nürnberger in [4]. Nürnberger and Sommer ([4, 5, 10]) have characterised those finite-dimensional Z-subspaces of $C([0, 1])$ for which there is a continuous selection. The necessity of their conditions follow easily from the lemmas of this paper.

The results concerning continuous selections which have been obtained to date suggest that it may be possible to obtain a complete description of those X and those finite-dimensional subspaces M of $C(X)$ for which there is a continuous selection. However the condition that M be a Z-subspace cannot simply be omitted from our theorem. For, if X is a disjoint union of X_1 and X_2 and the restriction of $M \subseteq C(X)$ to X_2 is zero, then any continuous selection for the restriction $M | X_1 \subseteq C(X_1)$ yields a continuous selection for M—and this entails no restriction upon X_2. Nor can the distinction between the metrisable and non-metrisable cases be eliminated. In the final section an example is given of a two-dimensional Z-subspace of $C(I_A)$ for which there is a continuous selection.

2. A Necessary Condition for the Existence of a Continuous Selection

The argument depends upon the following basic Lemma 1 concerning continuous selections for metric projections which originated in [3, Lemma 2.2] and, in the form stated here, is contained in [1, Lemma 2.6]. We use the notations $Z(f) = f^{-1}(0)$ for $f \in C(X)$ and $Z(A) = \bigcap \{Z(f): f \in A\}$ for $A \subseteq C(X)$.

Lemma 1. *Let M be a finite-dimensional subspace of $C(X)$ and $s: C(X) \to M$ a continuous selection for M. If $f \in C(X)$, $\|f\| = 1$ and $0 \in P(f)$ then for each $p \in P(f)$*

$$\{x: s(f)(x) \geqslant p(x)\} \text{ is a neighbourhood of } f^{-1}(1) \cap Z(P(f)),$$

$$\{x: s(f)(x) \leqslant p(x)\} \text{ is a neighbourhood of } f^{-1}(-1) \cap Z(P(f)).$$

Henceforth we will suppose that M is a finite-dimensional Z-subspace of $C(X)$ with $\dim M = n \geqslant 2$, and that there exists a continuous selection $s: C(X) \to M$ for M. If M is Chebyshev then the conclusion of the theorem is given by Mairhuber's theorem. So it will be supposed that M is not Chebyshev.

The analysis begins with a rehearsal of some of the elementary facts concerning Chebyshev and k-Chevyshev subspaces of $C(X)$. If N is any subspace of $C(X)$ of finite dimension n then the following statements are equivalent.

(1) N is a Chebyshev subspace of $C(X)$.

(2) Each non-zero function in N has at most $n-1$ distinct zeros in X.

(3) If $y_1, \ldots, y_n$ are distinct points of X then the restrictions to N of the evaluation functionals at $y_1, \ldots, y_n$ are linearly independent points of the dual space of N.

(4) If $y_1,\dots, y_n$ are distinct points of X and

$$\alpha_1 p(y_1) + \cdots + \alpha_n p(y_n) = 0$$

for all $p \in N$ then $\alpha_1 = \cdots = \alpha_n = 0$.

The equivalence of (1) and (2) is the classical result of Haar. The equivalence of (2), (3) and (4) is elementary.

If $A \subseteq X$ and N has the property that each non-zero function in N has at most $n-1$ distinct zeros in A it will be said that N satisfies the Haar condition on A.

Finite-dimensional subspaces of $C(X)$ which are k-Chebyshev have been characterized by Rubinshtein ([6], or v. [8]). The characterization can be stated conveniently in the form: a subspace M of $C(X)$ of finite dimension n is k-Chebyshev if and only if for any $n-k$ distinct points $y_1,\dots, y_{n-k}$ of X the restrictions to M of the evaluation functionals at $y_1,\dots, y_{n-k}$ are linearly independent.

The subspace M is not Chebyshev, so let $p_0 \in M$, $\|p_0\| = 1$, be a function with at least n distinct zeros $y_1,\dots, y_n$ in X. Then the restrictions to N of the evaluation functionals at $y_1,\dots, y_n$ are linearly dependent. If we choose a minimal linearly dependent subset we obtain $r \geqslant 1$ points $x_1,\dots, x_r$ (from amongst $y_1,\dots, y_n$) and non-zero constants $\alpha_1,\dots, \alpha_r$ such that

$$\alpha_1 p(x_1) + \cdots + \alpha_r p(x_r) = 0$$

for all $p \in M$. Let $\sigma_i = \operatorname{sgn} \alpha_i$, $i = 1,\dots, r$, and let

$$N = \{p \in M\colon p(x_1) = \cdots = p(x_r) = 0\}.$$

By the minimality property of $x_1,\dots, x_r$

$$\dim N = n + 1 - r.$$

Note that if $p \in N\backslash\{0\}$ has $n+1$ distinct zeros in X then it has $n+1-r$ distinct zeros in $X\backslash\{x_1,\dots, x_r\}$ and N does not satisfy the Haar condition on $X\backslash\{x_1,\dots, x_r\}$.

It is a well-known fact that if $f \in C(X)$, $\|f\| = 1$ and $f(x_i) = \sigma_i$ for $i = 1,\dots, r$ then $d(f, M) = \|f\| = 1$ and $0 \in P(f) \subseteq N$. For, if $p \in M$ and $\|f - p\| \leqslant 1$, then $\sigma_i p(x_i) \geqslant 0$ for $i = 1,\dots, r$ and

$$0 \leqslant |\alpha_1|\, \sigma_1 p(x_1) + \cdots + |\alpha_r|\, \sigma_r p(x_r) = \alpha_1 p(x_1) + \cdots + \alpha_r p(x_r) = 0,$$

which is only possible if $p \in N$, and then

$$\|f - p\| \geqslant |f(x_i) - p(x_i)| = 1.$$

Let $q(x) = \sup\{|p(x)|: p \in N, \|p\| \leqslant 1\}$. By the compactness and consequent equicontinuity of $\{p \in N: \|p\| \leqslant 1\}$ the function q is continuous, that is, $q \in C(X)$, and $\|q\| = 1$. If f satisfies the further condition

$$-1 + q(x) \leqslant f(x) \leqslant 1 - q(x)$$

for all $x \in X$ (there are such f) then $\{p \in N: \|p\| \leqslant 1\} \subseteq P(f)$. This is an elementary proof of the fact that (1) implies (2) and of one half of Rubinshtein's result. It is the appropriate preliminary to

LEMMA 2. *N satisfies the Haar condition on $X \backslash \{x_1, \dots, x_r\}$.*

Proof. Suppose not. Then there exist distinct points $y_1, \dots, y_s$ $(1 \leqslant s \leqslant n + 1 - r)$ of $X \backslash \{x_1, \dots, x_r\}$ and non-zero constants $\beta_1, \dots, \beta_s$ such that

$$\beta_1 p(y_1) + \cdots + \beta_s p(y_s) = 0$$

for all $p \in N$. Let

$$N' = \{p \in N: p(y_1) = \cdots = p(y_s) = 0\}.$$

Then $\dim N' \geqslant 1$. Let ε denote either 1 or -1. If

$$\begin{aligned} &f_\varepsilon \in C(X), && \|f_\varepsilon\| = 1, \\ &f_\varepsilon(x_i) = \sigma_i, && i = 1, \dots, r, \\ &f_\varepsilon(y_j) = \varepsilon \operatorname{sign} \beta_j, && j = 1, \dots, s, \end{aligned} \tag{5}$$

then $0 \in P(f) \subseteq N'$ (by a repetition of the argument preceding the statement of the lemma). Now we will construct functions f_ε ($\varepsilon = 1$ and -1) satisfying (5) and such that

$$\{p \in N': \|p\| \leqslant 1\} \subseteq P(f_1) = P(f_{-1}) \subseteq N'.$$

Let $q'(x) = \sup\{|p(x)|: p \in N', \|p\| \leqslant 1\}$. Then $q'(x) \in C(X)$, $\|q'\| = 1$ and $\{x_1, \dots, x_r, y_1, \dots, y_s\} \subseteq Z(q')$. Let W be a closed set such that

$$\{y_1, \dots, y_s\} \subseteq \operatorname{int} W \subseteq W \subseteq \{x \in X: 2q'(x) \leqslant 1\} \backslash \{x_1, \dots, x_s\}.$$

Then there exist functions f_ε ($\varepsilon = 1$ and -1) satisfying (5) and such that

$$-1 + q'(x) \leqslant f_\varepsilon(x) \leqslant 1 - q'(x) \quad \text{for} \quad x \in X, \tag{6}$$

$$-1 + 2q'(x) \leqslant f_\varepsilon(x) \leqslant 1 - 2q'(x) \quad \text{for} \quad x \in W, \tag{7}$$

$$f_1(x) = f_{-1}(x) \quad \text{for} \quad x \notin W. \tag{8}$$

Then, by (5) and (6), for $\varepsilon = 1$ or -1,

$$\{p \in N' : \|p\| \leqslant 1\} \subseteq P(f_\varepsilon) \subseteq N'.$$

Suppose that $p \in P(f_\varepsilon)$. If $x \notin W$ then by (8)

$$|f_{-\varepsilon}(x) - p(x)| = |f_\varepsilon(x) - p(x)| \leqslant 1.$$

Now $\|p\| \leqslant 2$ so that $|p(x)| \leqslant 2q'(x)$ for all $x \in X$. Therefore, by (7), if $x \in W$ then $|f_{-\varepsilon}(x) - p(x)| \leqslant 1$. This proves that $p \in P(f_{-\varepsilon})$. It now follows that $P(f_1) = P(f_{-1})$.

At this point in the argument the fact that M is a Z-subspace is used repeatedly. If $p \in N'$, $\|p\| = 1$ then $Z(p)$ has empty interior and $\{p, -p\} \subseteq P(f_1)$. Therefore, by Lemma 1, applied to f_1 and to both p and $-p$ in $P(f_1)$, $\{x: \sigma_i s(f_1)(x) \geqslant |p(x)|\}$ is a neigbourhood of x_i for $i = 1,..., r$, and it follows that $s(f_1) \neq 0$. Now, by Lemma 1 applied first to f_1 and $p = s(f_{-1}) \in P(f_1)$ and then to f_{-1} and $p = s(f_1) \in P(f_{-1})$ it follows that $\{x: s(f_1)(x) = s(f_{-1})(x)\}$ is a neighbourhood of $\{x_1,..., x_r\}$. Therefore $s(f_1) = s(f_{-1})$. Now by Lemma 1 applied to f_1 and $p = 0 \in P(f_1)$ and to f_{-1} and $p = 0 \in P(f_{-1})$ it follows that $\{x: s(f_1)(x) \leqslant 0 \leqslant s(f_1)(x)\}$ is a neighbourhood of $\{y_1,..., y_s\}$. Because M is a Z-subspace this implies that $s(f_1) = 0$, which is a contradiction. The proof of Lemma 2 is complete. The argument is a development of that of [1, Theorem 2.8 (ii)].

From Lemma 2 and the discussion preceding it we immediately obtain a result which in the case $X = [0, 1]$ is due to Sommer [10].

COROLLARY. *Each non-zero function $p \in M$ has at most n distinct zeros in X.*

There are now two cases to consider, according as M is or is not 1-Chebyshev. Thus the division into two cases for the purposes of the proof does not correspond to the division in the statement of the theorem: the proof is divided according to the properties of M, the theorem is stated in terms of properties of X.

3. CASE 1: M NOT A 1-CHEBYSHEV SUBSPACE

In this case by Rubinshtein's result we can choose the integer r, distinct points $x_1,..., x_r$, the numbers $\sigma_1,..., \sigma_r$ and the subspace N so that $1 \leqslant r \leqslant n-1$ and $\dim N = n + 1 - r \geqslant 2$.

The next lemma was suggested by the arguments of [4]. Note that if $x_{r+1},..., x_n$ are $n-r$ distinct points of $X \backslash \{x_1,..., x_r\}$ then, by Lemma 2, $\{p \in N: p(x_{r+1}) = \cdots = p(x_n) = 0\}$ is a one-dimensional subspace of N.

LEMMA 3. *Let* $x_{r+1},\dots,x_n$ *be* $n-r$ *distinct points of* $X\backslash\{x_1,\dots,x_r\}$ *and* $p_1\in N$ *a function such that* $p_1(x_{r+1})=\cdots=p_1(x_n)=0$. *Then there exist pairwise disjoint neighbourhoods* $V_1,\dots,V_r$ *of* $x_1,\dots,x_r$, *respectively, such that the function* p_1', *defined on* $\bigcup V_i$ *by* $p_1'(x)=\sigma_i p_1(x)$ *for* $x\in V_i$ *and* $i=1,\dots,r$, *is of constant sign on* $\bigcup V_i$.

Proof. Suppose that the conclusion does not hold. A contradiction to Lemma 2 will be obtained.

It may be supposed that $\|p_1\|=1$. Let $\sigma=(\sigma_{r+1},\dots,\sigma_n)\in\{1,-1\}^{n-r}$ (i.e. σ is an $(n-r)$-tuple each term of which is either 1 or -1). Choose a function $f\in C(X)$, $\|f\|=1$, such that

$$-1+p_1(x)\leqslant f(x)\leqslant 1+p_1(x) \qquad \text{for all } x\in X,$$

$$f(x_i)=\sigma_i \qquad \text{for } i=1,\dots,n.$$

Then $\{0,p_1\}\subseteq P(f)\subseteq N$. If $p\in P(f)$ then $\sigma_i p(x_i)\geqslant 0$ for $i=r+1,\dots,n$. Let $p_\sigma=s(f)$. By Lemma 1 applied to f and to both $0\in P(f)$ and $p_1\in P(f)$

$$\{x\colon \sigma_i p_\sigma(x)\geqslant \max\{0,\sigma_i p_1(x)\}\}$$

is a neighbourhood of x_i for $i=1,\dots,r$.

If $\tau=(-\sigma_{r+1},\ \sigma_{r+2},\dots,\sigma_n)$ then $\sigma_{r+1}p_\sigma(x_{r+1})\geqslant 0\geqslant\sigma_{r+1}p_\tau(x_{r+1})$ and therefore some convex combination of p_σ and p_τ will be zero at x_{r+1}. It now follows that by repeating this process we can obtain a function p_2 in the convex hull of $\{p_\sigma\colon\sigma\in\{1,-1\}^{n-r}\}$ such that $p_2(x_{r+1})=\cdots=p_2(x_n)=0$. Also, for each $i=1,\dots,r$, the set

$$V_i=\{x\colon \sigma_i p_2(x)\geqslant\max\{0,\sigma_i p_2(x)\}\}$$

is a neighbourhood of x_i. On the assumption that p_1' takes both positive and negative values on any union of pairwise disjoint neighbourhoods of $x_1,\dots,x_r$, it follows that p_1 and p_2 are linearly independent. Therefore $\{p\in N\colon p(x_{r+1}),\dots,p(x_n)=0\}$ is of dimension at least two, and this contradicts Lemma 2.

It follows from the fact that N satisfies the conclusions of Lemmas 2 and 3, and is of dimension at least two, that the space X (which has no isolated points) must be homeomorphic to a subspace of a circle. The proof that this is so (Lemma 5) is modelled on the proof of Mairhuber's theorem due to Schoenberg and Yang. However we need an additional simple topological lemma.

LEMMA 4. *Let* Y *be a compact Hausdorff space and* $\varphi\colon Y\to[0,1]$ *a continuous mapping.*

(a) *Suppose that* $\varphi^{-1}(0)=\{y_1,\dots,y_s\}$ *is a finite set of* s *points* $(s\geqslant 1)$

and that the restriction $\varphi \mid (Y \backslash \{y_1,..., y_s\})$ *is injective. Then there exist pairwise disjoint open and closed sets* $U_1,..., U_s$ *such that* $Y = \bigcup U_i$ *and* $y_i \in U_i$ *for* $i = 1,..., s$. *Consequently* Y *is homeomorphic to a subspace of* $[0, 1]$.

(b) *Suppose that* $\varphi(Y) = [0, 1]$, *that* $\varphi^{-1}(\{0, 1\})$ *is a finite set of non-isolated points and that the restriction* $\varphi \mid (Y \backslash \varphi^{-1}(\{0, 1\}))$ *is injective. Then* φ *is a homeomorphism.*

Proof. (a) If $s = 1$ there is nothing to prove. Suppose that $s > 1$. If $s - 1$ of the points $y_1,..., y_s$ are isolated then again there is nothing to prove. Therefore suppose that y_1 and y_2 are not isolated. Let V_1 be a closed neighbourhood of y_1 not containing any of $y_2,..., y_s$. If V_1 is open and closed let $U_1 = V_1$. If V_1 is not open and closed let $\tau = \min \varphi(\text{Fr } V_1)$ (where Fr V_1 denotes the frontier of V_1) and choose $y \notin V_1$ so that $0 < \varphi(y) < \tau$ (this is possible because y_2 is not isolated). Then $U_1 = \varphi^{-1}([0, \varphi(y)]) \cap V_1$ is open and closed in Y. The conclusion now follows by induction on s.

(b) It is only necessary to show that $\varphi^{-1}(0)$ and $\varphi^{-1}(1)$ are single points. Suppose on the contrary that $\varphi(y_1) = \varphi(y_2) = 0$ and $y_1 \neq y_2$. Then, as in the proof of (a), there is an open and closed neighbourhood U_1 of y_1 such that $U_1 \cap \varphi^{-1}(\{0, 1\}) = \{y_1\}$. Then $\varphi(U_1) \backslash \{0\}$, $\varphi(Y \backslash U_1) \backslash \{0\}$ is a disconnection of $(0, 1]$, which is impossible. This completes the proof.

In Lemma 5 it will be assumed only that the conclusions of Lemmas 2 and 3 are satisfied.

LEMMA 5. *Let* N, *a finite-dimensional subspace of* $C(X)$ *with* $\dim N = k \geqslant 2$, *distinct points* $x_1,..., x_r$ *of* X $(r \geqslant 1)$ *and* $\sigma_1,..., \sigma_r$ *in* $\{1, -1\}$ *have the three properties*

(i) $p(x_1) = \cdots = p(x_r) = 0$ *for all* $p \in N$,

(ii) N *satisfies the Haar condition on* $X \backslash \{x_1,..., x_r\}$,

(iii) *if* $p \in N$ *has* $n - r$ *distinct zeros in* $X \backslash \{x_1,..., x_r\}$ *then for some pairwise disjoint neighbourhoods* $V_1,..., V_r$ *of* $x_1,..., x_r$, *respectively, the function* p', *defined on* $\bigcup V_i$ *by* $p'(x) = \sigma_i p(x)$ *for* $x \in V_i$ *and* $i = 1,..., r$, *has constant sign on* $\bigcup V_i$.

Then X *is homeomorphic either to the union of a circle and a finite set of isolated points or to a subspace of a circle.*

Proof. If $x_1,..., x_r$ are all isolated points of X then by (ii) and Mairhuber's theorem $X \backslash \{x_1,..., x_r\}$ is homeomorphic to a subset of a circle and the conclusion of the lemma follows. If not all of $x_1,..., x_r$ are isolated then those which are isolated can be ignored. So we may assume that none of $x_1,..., x_r$ are isolated.

First consider the case $\dim N = k = 2$. It will be proved that X is homeomorphic to a subspace of an interval. Let $N = \mathrm{sp}\{p_1, p_2\}$ and define $\varphi\colon X\backslash\{x_1,\dots,x_r\} \to S^1$ by

$$\varphi(x) = \left(\frac{p_1(x)}{(p_1(x)^2 + p_2(x)^2)^{\frac{1}{2}}}, \frac{p_2(x)}{(p_1(x)^2 + p_2(x)^2)^{\frac{1}{2}}}\right). \tag{9}$$

Then φ is well defined (because of (ii)) and is continuous. For $z \in R^2\backslash\{0\}$ let L_z denote the line (one-dimensional linear subspace of the plane R^2) through z. Properties (ii) and (iii) are equivalent to

(ii)′ if L is a line through the origin then $\varphi^{-1}(L)$ is at most one point of $X\backslash\{x_1,\dots,x_r\}$, and

(iii)′ for each $x \in X\backslash\{x_1,\dots,x_r\}$ there exist pairwise disjoint neighbourhoods $V_1,\dots,V_r$ of $x_1,\dots,x_r$, respectively, such that $\bigcup_{i=1}^r \sigma_i\varphi(V_i\backslash\{x_i\})$ is contained in one of the closed half-planes determined by $L_{\varphi(x)}$.

Let Σ be the set of points $z \in S^1$ which are such that there exists an i and a net (x_α) in $X\backslash\{x_i\}$ convergent to x_i such that $z = \lim_\alpha \sigma_i\varphi(x_\alpha)$. The set Σ consists of either one or two points: the points $x_1,\dots,x_r$ are not isolated so Σ contains at least one point; if Σ were to contain three points then there would be a line L through one of them strictly separating the other two, then for some line $L_{\varphi(x)}$ close to L a contradiction to (iii)′ would be obtained. It is now necessary to consider separately the two cases of Σ having one and two points.

Suppose $\Sigma = \{z_1, z_2\}$ where $z_1 \neq z_2$. Then it follows from (iii)′ that no line $L_{\varphi(x)}$ can separate z_1 and z_2 (and by (ii)′ this implies that $z_1 \neq -z_2$) and then, further, that $\varphi(x) \neq z_1$, $\varphi(x) \neq z_2$ for all $x \in X\backslash\{x_1,\dots,x_r\}$. Let A be the minor closed arc of S^1 from z_1 to $-z_2$ (not containing $-z_1$ or z_2) and let $B = -A$. Then $\varphi(X\backslash\{x_1,\dots,x_r\}) \subseteq A \cup B$ and $\varphi^{-1}(A)$, $\varphi^{-1}(B)$ are disjoint sets each of which is open and closed in their union $X\backslash\{x_1,\dots,x_r\}$. Also, for the closure in X,

$$\varphi^{-1}(A)^- \subseteq \varphi^{-1}(A) \cup \{x_1,\dots,x_r\},$$

$$\varphi^{-1}(B)^- \subseteq \varphi^{-1}(B) \cup \{x_1,\dots,x_r\}.$$

The set $\varphi^{-1}(A)$ has the further property that if $x_i = \lim x_\alpha$, where (x_α) is a net in $\varphi^{-1}(A)$ then $\varphi(x_\alpha)$ is convergent to z_1 if $\sigma_i = 1$ and to $-z_2$ if $\sigma_i = -1$. There is a corresponding property of $\varphi^{-1}(B)$.

If $\varphi^{-1}(B)$ is empty then φ extends to a unique continuous mapping of X onto A (we use the same symbol φ) such that $\varphi^{-1}(z_1) = \{x_i\colon \sigma_i = 1\}$, $\varphi^{-1}(-z_2) = \{x_i\colon \sigma_i = -1\}$. If $\varphi(X) = A$ then the conclusion follows from Lemma 4(b). If $\varphi(X) \neq A$ then there exist disjoint closed subarcs A_1 and A_2 of A such that $\varphi(X) = A_1 \cup A_2$. The conclusion follows by applying Lemma 4(a) to each of $\varphi \mid \varphi^{-1}(A_j) \to A_j$ $(j = 1, 2)$. In the same way the conclusion follows if $\varphi^{-1}(A)$ is empty.

Suppose that $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$ are both non-empty. If $\varphi(x) \in B$ then $-\varphi(x) \in A\backslash\varphi(X)$ by (ii)$'$. Let A_1, A_2 be disjoint closed subarcs of A such that $z_1 \in A_1$, $-z_2 \in A_2$ and $\varphi(X) \cap A \subseteq A_1 \cup A_2$. Similarly let B_1 and B_2 be disjoint closed subarcs of B such that $z_2 \in B_2$, $-z_1 \in B_1$ and $\varphi(X) \cap B \subseteq B_1 \cup B_2$. Then $\varphi^{-1}(A_1)$, $\varphi^{-1}(A_2)$, $\varphi^{-1}(B_1)$ and $\varphi^{-1}(B_2)$ are pairwise disjoint sets each open and closed in their union $X\backslash\{x_1,..., x_r\}$. Furthermore, for their closures in X we have

$$(\varphi^{-1}(A_1)^-\backslash\varphi^{-1}(A_1)) \cup (\varphi^{-1}(B_2)^-\backslash\varphi^{-1}(B_2)) \subseteq \{x_i: \sigma_i = 1\},$$
$$(\varphi^{-1}(A_2)^-\backslash\varphi^{-1}(A_2)) \cup (\varphi^{-1}(B_1)^-\backslash\varphi^{-1}(B_1)) \subseteq \{x_i: \sigma_i = -1\}.$$

The restrictions of φ to each of $\varphi^{-1}(A_1)$, $\varphi^{-1}(A_2)$, $\varphi^{-1}(B_1)$ and $\varphi^{-1}(B_2)$ extend to continuous mappings of the closures into the arcs A_1, A_2, B_1 and B_2 (such that the image of each x_i is an end point of the arc). Now we can apply Lemma 4(a) to each of these mappings. It follows that each of the subspaces $\varphi^{-1}(A)^-$ and $\varphi^{-1}(B)^-$ of X can be expressed as a union of at most r sets, each open and closed in the subspaces, containing one of $x_1,..., x_r$, and homeomorphic to a subset of an interval by a homeomorphism mapping the x_i to an end point of the interval. If an x_i belongs to two of these sets then their union is homeomorphic to a subset of an interval. Thus X is expressed as a union of r open and closed sets each homeomorphic to a subspace of an interval.

Now suppose that $\Sigma = \{z_1\}$. If z_1 or $-z_1$ is the image of a point in $X\backslash\{x_1,..., x_r\}$ then denote that point (there is only one, by (ii)$'$) by x_{r+1} and put $\sigma_{r+1} = 1$ if $\varphi(x_{r+1}) = z_1$ and $\sigma_{r+1} = -1$ if $\varphi(x_{r+1}) = -z_1$. Thus at the expense of increasing r by one we may suppose that z_1 and $-z_1$ are not in $\varphi(X\backslash\{x_1,..., x_r\})$. Let A and B be the two closed half circles between z_1 and $-z_1$. The argument is now concluded in the same way as in the previous case.

This completes the proof that in case $\dim N = k = 2$ the space X is homeomorphic to a subspace of an interval.

Now suppose that $\dim N > 2$. Let $y_1,..., y_{k-2}$ by any $k-2$ distinct points of $X\backslash\{x_1,..., x_r\}$ and let

$$N' = \{p \in N: p(y_1) = \cdots = p(y_{k-2}) = 0\}.$$

Then, by (ii), $\dim N' = 2$ and N' satisfies the hypotheses of the lemma (for $k = 2$) on any closed subset X' of $X\backslash\{y_1,..., y_{k-2}\}$. Therefore, by what has been proved, any such X' is homeomorphic to a subspace of an interval. It now follows by repeated application of a lemma of Schoenberg and Yang ([7, Lemma 1], or [8, p. 219]) that X is homeomorphic to a subspace of a union of a circle and a finite set of isolated points.

We close this section by summarising as a proposition the achievement of the argument so far. The proposition follows directly from Lemmas 2, 3 and 5 and the fact that X cannot have isolated points.

PROPOSITION 1. *If there exists a Z-subspace of $C(X)$, of finite dimension at least two, which is not 1-Chebyshev, and for which there is a continuous selection, then X is homeomorphic to a subspace of a circle (and so is metrisable).*

4. CASE 2: M A 1-CHEBYSHEV, BUT NOT CHEBYSHEV SUBSPACE

PROPOSITION 2. *If there exists a Z-subspace of $C(X)$, of finite dimension at least two, which is 1-Chebyshev but not Chebyshev and for which there is a continuous selection, then X is homeomorphic either to a circle or to a subpace of an interval with split points.*

Intervals with split points were defined in the Introduction.

Suppose that M satisfies the hypotheses of the proposition and let $s: C(X) \to M$ be a continuous selection for M. Then, by the results of Haar and Rubinshtein, if $x_1,..., x_n$ are distinct points of X and there exist $\alpha_1,..., \alpha_n$, not all zero, such that

$$\alpha_1 p(x_1) + \cdots + \alpha_n p(x_n) = 0$$

for all $p \in M$, then $\alpha_1,..., \alpha_n$ are all non-zero. Furthermore there does exist at least one such set of n points $x_1,..., x_n$. In this situation the subspace

$$N = \{p \in M: p(x_1) = \cdots = p(x_n) = 0\}$$

is of dimension one. If $N = \text{sp}\{p\}$, $\|p\| = 1$ and $\sigma_i = \text{sgn}\, \alpha_i$, for $i = 1,..., n$, then there exists $f \in C(X)$, $\|f\| = 1$, such that $f(x_i) = \sigma_i$ for $i = 1,..., n$ and $\{-p, p\} \subseteq P(f) \subseteq N$. Therefore $s(f)$ is a multiple of p and, by Lemma 1, there exist disjoint neighbourhoods $V_1,..., V_n$ of $x_1,..., x_n$, respectively, such that the function p' defined on $\bigcup V_i$ by $p'(x) = \sigma_i p(x)$ for $x \in V_i$ and $i = 1,..., n$, is of constant sign on $\bigcup V_i$. Also, by Lemma 2, p has no zero in $X \backslash \{x_1,..., x_n\}$. The proposition will now follow from

LEMMA 6. *Suppose that there exists a Z-subspace M of $C(X)$, with $\dim M = n \geqslant 2$, and with the properties:*

(i) *Each non-zero function in M has at most n distinct zeros in X.*

(ii) *There does exist a non-zero function in M with n distinct zeros in X.*

(iii) *If $p \in M$, $p \neq 0$, has n distinct zeros $x_1,..., x_n$ and*

$$\alpha_1 q(x_1) + \cdots + \alpha_n q(x_n) = 0$$

for all $q \in M$, *where* $\alpha_1,\ldots,\alpha_n$ *are not all zero, then* $\alpha_1,\ldots,\alpha_n$ *are all non-zero and there exist pairwise disjoint neighbourhoods* $V_1,\ldots,V_n$ *of* $x_1,\ldots,x_n$, *respectively such that the function* p' *defined on* $\bigcup V_i$ *by* $p'(x)=\sigma_i p(x)$ ($\sigma_i = \operatorname{sgn} \alpha_i$) *for* $x \in V_i$ *and* $i = 1,\ldots, r$, *is of constant sign on* $\bigcup V_i$.

Then X is homeomorphic either to a circle or to a subspace of an interval with split points.

Proof. First consider the case $n=2$. Let $M=\operatorname{sp}\{p_1, p_2\}$. Again define $\varphi: X\to S^1$ by (9) (φ is well defined by (iii), which implies that $Z(M)=\varnothing$). Then conditions (i), (ii) and (iii) are equivalent to:

(i)′ If L is a line (through the origin) in R^2 then $\varphi^{-1}(L)$ consists of at most two points of X.

(ii)′ For some line L the set $\varphi^{-1}(L)$ contains two points.

(iii)′ If $z=\varphi(x_1)=\varphi(x_2)$ and $x_1\neq x_2$ then there exist disjoint neighbourhoods V_1, V_2 of x_1, x_2, respectively, such that $\varphi(V_1)$ and $\varphi(V_2)$ are separated by L_z, the line through z; if $z=\varphi(x_1)=-\varphi(x_2)$ then there exist disjoint neighbourhoods V_1, V_2 of x_1, x_2, respectively, such that $\varphi(V_1)\cup\varphi(V_2)$ is contained in one of the half-planes determined by L_z.

If $z=\varphi(x_1)=\varphi(x_2)$ and $x_1\neq x_2$ then, by (i)′, $-z\notin\varphi(X)$. If $z=\varphi(x_1)=-\varphi(x_2)$ then by (i)′ and the second part of (iii)′ (and a compactness argument) $\varphi(X)$ is not a neighbourhood in S^1 of z. By (ii)′ one of these situations must occur, and therefore $\varphi(X)\neq S^1$. Therefore we can obtain from φ a continuous mapping $\Psi: X\to[0, 1]$ with the properties:

(i)″ For each $t\in[0, 1]$ the set $\Psi^{-1}(t)$ is at most two points.

(iii)″ If $t=\Psi(x)=\Psi(y)$ and $x\neq y$ then there exist disjoint neighbourhoods V_1, V_2 of x, y, respectively, such that t separates $\Psi(V_1)$ and $\Psi(V_2)$.

Let $A=\{t\in[0, 1]: \Psi^{-1}(t)$ is a set of two points$\}$. Now we define a mapping Ψ': $X\to I_A$. If $\Psi^{-1}(\Psi(x))=\{x\}$ let $\Psi'(x)=(\Psi(x), 0)$. If $\Psi(x)=\Psi(y)$ for some $y\neq x$ then with V_1, V_2 as in (iii)″ let $\Psi'(x)=(\Psi(x), 0)$ if $\varphi(V_1)\subseteq[0, \Psi(x)]$ and let $\Psi'(x)=(\Psi(x), 1)$ if $\varphi(V_1)\subseteq[\Psi(x), 1]$. Then it is easy to verify that Ψ' is continuous. The mapping Ψ' is also injective and so the lemma is proved in the case $n=2$.

Now suppose that $\dim M=n>2$. Let $y_1,\ldots, y_{n-2}$ be any $n-2$ distinct point of X and put

$$N=\{p\in M: p(y_1)=\cdots=p(y_{n-2})=0\}.$$

Then it follows from (iii) that $\dim N=2$. Furthermore, if X' is a non-empty closed subset of X with no isolated points and $X'\subseteq X\backslash\{y_1,\ldots, y_{n-2}\}$ then either N satisfies the Haar condition on X' or it satisfies the conditions of the lemma for $n=2$. Therefore by Mairhuber's theorem and the first case considered X' is homeomorphic to a subspace of an interval with split points. The space X has no isolated points and therefore X' may be any proper closed subset of X without isolated points.

Suppose that X is not connected. If $X = X_1 \cup X_2$ is a disconnection of X then each of X_1, X_2 is homeomorphic to a subspace of an interval with split points, and therefore X is also.

Now suppose that X is connected. Let U be any proper open subset of X. Choose a non-empty open V such that $V^- \subseteq U$ and $X' = X \backslash V$ has no isolated points. Let $\Psi: X' \to I_A$ be continuous and injective. If $x \in X \backslash V^-$ then the component W of X' which contains x is not a single point (cf. [7, Lemma 1]). Therefore $\Psi(W)$ is a non-trivial connected subset of I_A. This can only happen if $\Psi(W)$ is an interval of I_A, the interior of which contains no point of $A \times \{0, 1\}$. It now follows that $\Psi(X \backslash U) \subseteq I \times \{0\}$ and the composite mapping $X \backslash U \to I_A \to I$ (in which the second mapping is the natural projection) is continuous and injective. Thus $X \backslash U$ is homeomorphic to a subset of an interval. It now follows by [7, Lemma 1] that X is homeomorphic to a subset of a circle. The proof of Lemma 6 is complete.

It remains to consider when the space I_A is metrisable. The next Lemma is an extension of [2, p. 104, (v)]. The natural projection of I_A onto I will be denoted $\pi: I_A \to I$.

LEMMA 7. *Let Y be a topological space with a countable base for its topology. If $\varphi: Y \to I_A$ is a continuous mapping then $\varphi(Y) \cap A \times \{1\}$ is countable.*

Proof. Let $\{U_n : n = 1, 2, \ldots\}$ be a base for the topology of Y. If $\varphi(y) \in A \times \{1\}$ then $\{z \in Y : \varphi(z) \geqslant \varphi(y)\}$ is a neighbourhood of y and so $y \in U_n \subseteq \{z \in Y : \varphi(z) \geqslant \varphi(y)\}$ for some n. Let J be the set of integers n such that for some $y \in \varphi^{-1}(A \times \{1\})$, $y \in U_n \subseteq \{z \in Y : \varphi(z) \geqslant \varphi(y)\}$. Then $\bigcup \{U_n : n \in J\} \supseteq \varphi^{-1}(A \times \{1\})$. If $n \in J$ and y, z are both points of $U_n \cap \varphi^{-1}(A \times \{1\})$ then $\varphi(y) = \varphi(z)$, and therefore $\varphi(U_n) \cap (A \times \{1\})$ is a single point. This proves the lemma.

PROPOSITION 3. *Let $X = I_A$ be an interval with split points. Then the following conditions are equivalent.*

(1) *X is metrisable.*

(2) *The set A is countable.*

(3) *X is homeomorphic to a subspace of I.*

Proof. X is a separable compact Hausdorff space and if it is metrisable it has a countable base for its topology. Therefore the implication (1) $\Rightarrow$ (2) follows from Lemma 7 applied to the identity mapping $X \to I_A$.

Suppose that $A = \{x_n : n = 1, 2, \ldots\}$ is countable. Define $\varphi : I_A \to I$ by

$$\varphi((x, t)) = \frac{x}{2} + \sum_{x_n < x} \frac{1}{2^{n+1}}, \qquad \text{if} \quad t = 0,$$
$$= \frac{x}{2} + \sum_{x_n \leqslant x} \frac{1}{2^{n+1}}, \qquad \text{if} \quad t = 1.$$

It is easy to verify that φ is continuous and injective. This proves that (2) ⇒ (3). Condition (3) ⇒ (1) requires no proof.

The theorem now follows from Propositions 1, 2 and 3, and Mairhuber's theorem.

5. An Example on Intervals with Split Points

Let A be a non-empty subset of I containing neither 0 nor 1. Then the interval with split points I_A has no isolated points. Let $g_1, g_2 \in C(I)$ be a Chebyshev system, i.e., $\mathrm{sp}\{g_1, g_2\}$ is a Chebyshev subspace of $C(I)$. Note that if t is a point of the open interval $(0, 1)$ and $g \in \mathrm{sp}\{g_1, g_2\}$, $g \neq 0$, $g(t) = 0$ then g changes sign at t. Let $\pi : I_A \to I$ be the natural projection and let $M = \mathrm{sp}\{g_1 \pi, g_2 \pi\}$. Let P denote the metric projection of $C(I_A)$ onto M.

Proposition 4. *The subspace M of $C(I_A)$ is a non-Chebyshev Z-subspace of $C(I_A)$ for which there is a continuous selection.*

Proof. That M is a Z-subspace and non-Chebyshev is immediate. If there is a continuous selection for M then it is unique and is identified by Lemma 1 (see [1, Lemma 2.6, Theorem 2.8]). In the light of this we define a selection $s : C(I_A) \to M$ for the metric projection in the following way. Consider $f \in C(I_A)$. If $P(f)$ is a single point let $s(f)$ be that single point. Suppose that $P(f)$ is not a single point. Choose any p in the relaive interior of the convex set $P(f)$. Then the function 0 is in the relative interior of $P(f - p)$ and $P(f - p) \neq \{0\}$. If $x \in I_A$ and $|(f - p)(x)| = \|f - p\|$ then $x \in Z(P(f - p))$ (this is well known, and explicit in [1, Lemma 2.2]). If $q \in M$ then $Z(q)$ is either a single point of $(I \backslash A) \times \{0\}$ or a pair $\{(t, 0), (t, 1)\}$ of points for some $t \in A$. If the set $Z(P(f - p))$ were a single point then a standard simple argument would show that $0 \notin P(f - p)$. Therefore $Z(P(f - P))$ is a pair $\{(t, 0), (t, 1)\}$ of points and $P(f - p)$ is of dimension one. Furthermore one must have

$$(f - p)((t, 0)) = -(f - p)((t, 1)) = \pm \|f - p\|$$

(for otherwise, again, $0 \notin P(f - p)$).

Let $q \in M$ be a non-zero function such that $q((t,0)) = q((t,1)) = 0$ and such that q is of the same sign as $f - p$ in a neighbourhood of $(t, 1)$ (and also of $(t, 0)$). This defines q to within a positive multiple. Then $P(f-p) = \{\lambda q : \alpha \leqslant \lambda \leqslant \beta\}$ for some real numbers α and β. Define $s(f) = \beta q + p$. It is trivial to show that this defines $s(f)$ unambiguously.

The mapping $s: C(I_A) \to M$ is a selection for the metric projection onto M and it has the properties

$$s(f - p') = s(f) - p' \quad \text{for } p' \in M, \quad s(\lambda f) = \lambda s(f) \quad \text{for } \lambda \in R.$$

It remains to prove that s is continuous. The argument which follows is similar to that of [1, Lemma 3.9].

If $f \in C(I_A)$ and $P(f)$ is a single point then s is continuous at f because the metric projection is upper semi-continuous. Suppose that s is not continuous at some $f \in C(X)$. Then $P(f)$ is not a single point, and, by translation, we may suppose that the function 0 is in the relative interior of $P(f)$. As before let $P(f) = \{\lambda q : \alpha \leqslant \lambda \leqslant \beta\}$, where $q((t,0)) = q((t,1)) = 0$, and $f(x)\, q(x) \geqslant 0$ for x in some neighbourhoods of $(t, 0)$ and $(t, 1)$. We may also suppose that

$$f((t,1)) = -f((t,0)) = \|f\| = 1.$$

Let (f_n) be a sequence in $C(I_A)$ such that $f = \lim f_n$ but $(s(f_n))$ does not converge to $s(f) = \beta q$. Any cluster point of $(s(f_n))$ is in $P(f)$, so, by extracting a subsequence we may suppose that $s(f_n)$ is convergent to μq, where $\alpha \leqslant \mu < \beta$.

Let V_0, V_1 be open neighbourhoods in I_A of $(t, 0)$ and $(t, 1)$, respectively, such that $f(x) - \mu q(x) \geqslant \frac{3}{4}$ and $q(x) \geqslant 0$ for $x \in V_1$, and $f(x) - \mu q(x) \leqslant -\frac{3}{4}$ and $q(x) \leqslant 0$ for $x \in V_0$. Choose $\theta > 0$ so that $\theta < \min\{\beta - \mu, \|q\|^{-1}\}$. Then $(\mu + \theta)q$ is in the relative interior of $P(f)$ and

$$|f(x) - (\mu + \theta)\, q(x)| < \|f\| = 1$$

unless x is one of the points $(t, 0)$ or $(t, 1)$. Let

$$K = \max\{|f(x) - (\mu + \theta)\, q(x)| : x \notin V_0 \cup V_1\}.$$

Then $K < 1$. Choose n so that

$$\|(f_n - s(f_n)) - (f - \mu q)\| \leqslant \min\{\tfrac{1}{2}(1 - K), \tfrac{1}{4}\}.$$

Then $f_n(x) - s(f_n)(x) \geqslant \frac{1}{2}$ for $x \in V_1$, $f_n(x) - s(f_n)(x) \leqslant -\frac{1}{2}$ for $x \in V_0$.

Now we prove that

$$\|f_n - s(f_n) - \theta q\| \leqslant \|f_n - s(f_n)\|. \tag{10}$$

If $x \notin V_0 \cup V_1$ then

$$\begin{aligned}
&|(f_n - s(f_n) - \theta q)(x)| \\
&\quad \leqslant |f(x) - (\mu + \theta)\, q(x)| + \|(f_n - s(f_n)) - (f - \mu q)\| \\
&\quad \leqslant K + \tfrac{1}{2}(1 - K) \\
&\quad \leqslant \|f_n - s(f_n)\|.
\end{aligned}$$

If $x \in V_1$ then

$$\begin{aligned}
f_n(x) - s(f_n)(x) - \theta q(x) &\geqslant f_n(x) - s(f_n)(x) - 1 \\
&\geqslant -(f_n(x) - s(f_n)(x)), \\
f_n(x) - s(f_n)(x) - \theta q(x) &\leqslant f_n(x) - s(f_n)(x),
\end{aligned}$$

and therefore

$$|f_n(x) - s(f_n)(x) - \theta q(x)| \leqslant \|f_n - s(f_n)\|.$$

The same inequality holds in V_0. This proves (10). Consequently $s(f_n) + \theta q \in P(f_n)$. Thus $P(f_n)$ is one dimensional and $s(f_n) + \frac{1}{2}\theta q$ is in its relative interior. We must have

$$P(f_n - s(f_n) - \tfrac{1}{2}\theta q) = \{\lambda q : \alpha_n \leqslant \lambda \leqslant \beta_n\}$$

for some $\alpha_n \leqslant -\frac{1}{2}\theta$, $\beta_n \geqslant \frac{1}{2}\theta$. Now $f_n - s(f_n)$ has the same sign as q in neighbourhoods of $(t, 0)$ and $(t, 1)$. Therefore $s(f_n) = (s(f_n) + \frac{1}{2}\theta q) + \beta_n q$, which is a contradiction.

The proof of Proposition 4 is complete.

References

1. A. L. Brown, On continuous selections for metric projections in spaces of continuous functions, *J. Funct. Anal.* **8** (1971), 431–449. MR 45.5725, Zbl 224.41013.
2. A. L. Brown, Chebyshev subspaces of finite codimension in spaces of continuous functions, *J. Austral. Math. Soc. Ser. A* **26** (1978), 99–109.
3. A. J. Lazar, D. E. Wulbert, and P. D. Morris, Continuous selections for metric projections, *J. Funct. Anal.* **3** (1969) 193–216. MR 39.3288, Zbl 174.171.
4. G. Nürnberger, "Nonexistence of Continuous Selections for the Metric Projection," Institut für Angewandte Mathematik I, Der Universität, Erlangen–Nürnberg, West Germany.
5. G. Nürnberger and M. Sommer, Weak Chebyshev subspaces and continuous selections for the metric projection, *Trans. Amer. Math. Soc.* **238** (1978), 129–138.
6. G. Sh. Rubinshtein, On a method of investigation of convex sets, *Doklady Akad. Nauk SSSR* **102** (1955), 451–454. MR 17, p. 185, Zbl 70, p. 109.

7. I. J. SCHOENBERG, AND C. T. YANG, On the unicity of solutions of problems of best approximation, *Ann. Mat. Pura Appl.* **54** (1961), 1–12, MR 25.5324, Zbl 99, p. 46.
8. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces." Springer-Verlag, Berlin, 1970. MR 38.3677; 42.4937, Zbl 154.145; 197.386.
9. M. SOMMER, Continuous selections of the metric projections for 1-Chebyshev spaces, *J. Approx. Theory* 76 (1979), 46–53.
10. M. SOMMER, "Nonexistence of Continuous Selections of the Metric Projection for a Class of Weak Chebyshev Spaces," Institut für Angewandte Mathematik I, Der Universität, Erlangen–Nürnberg, West Germany.